Decidability of Diophantine satisfiability in theories close to IOpen

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42nd Weak Arithmetic Days, Karlovassi, Greece
September 26, 2023

Diophantine satisfiability

Definition (Diophantine satisfiability decision problem)

Let $L := \{0, \mathrm{s}, +, \cdot\}$ be the base language of arithmetic and let T be a theory in a language $L' \supseteq L$. Is

$$D_T := \left\{ (t(\bar{x}), u(\bar{x})) \;\middle|\; \begin{array}{l} t(\bar{x}), u(\bar{x}) \text{ are } L\text{-terms such that} \\ T \cup \{\exists \bar{x}\, t(\bar{x}) = u(\bar{x})\} \text{ is consistent} \end{array} \right\}$$

decidable?

Observation

$D_T = \{(t, u) \mid T \vdash \forall \bar{x}\, t \neq u\}^c$. Thus D_T is decidable if and only if the set of T-refutable Diophantine equations is decidable.

Current results

- D_Q is decidable where Q is Robinson arithmetic[1]
- D_T is undecidable for theories T which extend $I\Delta_0 + EXP$ (consequence of the MRDP theorem)[2]
- D_T is undecidable for theories T which extend IU_1^-[3]
- Decidability of D_{IOpen} where *IOpen* is theory of open induction over $\{0, \mathrm{s}, +, \cdot, \leq\}$ is long-standing open problem[4]
- We show Diophantine decidability of the theory of open induction over $\{0, \mathrm{s}, \mathrm{p}, +, \cdot\}$

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[1] Jeř16.
[2] GD82.
[3] Kay93.
[4] She64.

Outline

IOp

- Language $L_{\mathsf{p}} := \{0, \mathsf{s}, +, \cdot, \mathsf{p}\}$
- Base theory A: universal closures of

$$\begin{aligned} \mathsf{s}(x) &\neq 0 && (A_1) \\ \mathsf{p}(0) &= 0 && (A_2) \\ \mathsf{p}(\mathsf{s}(x)) &= x && (A_3) \\ x + 0 &= x && (A_4) \\ x + \mathsf{s}(y) &= \mathsf{s}(x + y) && (A_5) \\ x \cdot 0 &= 0 && (A_6) \\ x \cdot \mathsf{s}(y) &= x \cdot y + x && (A_7) \end{aligned}$$

- Induction axiom $I(\varphi(x, \bar{z}))$

$$\forall \bar{z}\ (\varphi(0, \bar{z}) \to \forall x\, (\varphi(x, \bar{z}) \to \varphi(\mathsf{s}(x), \bar{z})) \to \forall x\, \varphi(x, \bar{z}))$$

- $\mathrm{IOp} := A \cup \{ I(\varphi) \mid \varphi$ quantifer-free L_{p}-formula$\}$

IOp

Result by Shepherdson[5]

IOp is equivalent to A together with universal closures of

$$
\begin{aligned}
x = 0 \lor x = \mathrm{s}(\mathrm{p}(x)) && (B_1)\\
x + y = y + x && (B_2)\\
(x + y) + z = x + (y + z) && (B_3)\\
x + y = x + z \to y = z && (B_4)\\
x \cdot y = y \cdot x && (B_5)\\
x \cdot (y \cdot z) = (x \cdot y) \cdot z && (B_6)\\
x \cdot (y + z) = x \cdot y + x \cdot z && (B_7)
\end{aligned}
$$

and

$$dx = dy \to \bigvee_{i=0}^{d-1} (z + i) \cdot x = (z + i) \cdot y \ (C'_d) \quad \text{for } d \geq 2.$$

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[5] She67.

Related theories

- $AB := \{A_1, \dots, A_7, B_1, \dots, B_7\}$
- $AB^{\exists} := (AB \setminus \{A_2, A_3, B_1\}) \cup \left\{B_1^{\exists}\right\}$ where $B_1^{\exists}$ is the universal closure of
$$x = 0 \vee \exists y\, x = \mathsf{s}(y)$$
- $ABC_d := AB^{\exists} \cup \{\, C_d \mid d \geq 2\}$ where C_d is the universal closure of
$$dx = dy \rightarrow x = y$$

Theorem (Schmerl[6])

$D_{\mathsf{IOp}} = D_{AB} = D_{AB^{\exists}} = D_{ABC_d}$

Main Theorem

D_{IOp} *is decidable.*

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[6] Sch88.

Outline

Theories

Proof strategy

Decidability

Conclusion

Proof strategy

- By result from Schmerl it suffices to show decidability of $D_{AB^{\exists}}$
- Construct a specialized proof calculus $\mathcal{AB}$ operating on $\mathbb{Z}[V]$
- Show soundness and completeness of $\mathcal{AB}$ with respect to Diophantine satisfiability in $AB^{\exists}$
- Show decidability of $\mathcal{AB}$

Terms as polynomials

- Let V be the set of variables.
- To a term t we assign the polynomial $\text{poly}(t) \in \mathbb{N}[V]$ it evaluates to.
- To a $p \in \mathbb{N}[V]$ we assign a term $\underline{p}$ (by choosing a fixed ordering on V) such that

Lemma

For every term t we have $AB^{\exists} \vdash t = \underline{\text{poly}(t)}$.

Equations as polynomials

- For $p \in \mathbb{Z}[V]$ and a monomial m we write $[m]p$ for the coefficient of m in p.
- We set

$$p^+ := \sum_{m:[m]p>0} ([m]p)m \qquad p^- := -\sum_{m:[m]p<0} ([m]p)m.$$

Consider the additive cancellation axiom

$$x + y = x + z \to y = z \quad (B_4)$$

Lemma

Let t, u be terms and set $p := \mathrm{poly}(t) - \mathrm{poly}(u)$. Then $AB^{\exists} \vdash t = u \leftrightarrow \underline{p^+} = \underline{p^-}$

Calculus $\mathcal{AB}$

signed rule

Definition (signed polynomial)

$p \in \mathbb{Z}[V]$ is *positively signed* if all coefficients of p are non-negative and the constant coefficient is positive.
p is *negatively signed* if $-p$ is positively signed.
p is *signed* if it is positively or negatively signed

Consider the axiom A_1

$$\mathsf{s}(x) \neq 0$$

We translate this into an initial inference rule on polynomials

$$\frac{}{p}\ \textit{signed}$$

where $p \in \mathbb{Z}[V]$ is signed

Calculus $\mathcal{AB}$

zero-or-successor rule

Consider the axiom $B_1^{\exists}$, the universal closure of

$$x = 0 \lor \exists y\, x = \mathsf{s}(y)$$

In $AB^{\exists} \setminus \{B_1^{\exists}\}$, instead of considering all possible instances of $B_1^{\exists}$ it is enough consider variable instances:

Proposition

Let t be a term and let $x_1, \ldots, x_n$ be all its free variables. Then

$$AB^{\exists} \setminus \left\{B_1^{\exists}\right\}, B_1^{\exists}[x_1], \ldots, B_1^{\exists}[x_n] \vdash B_1^{\exists}[t]$$

Calculus $\mathcal{AB}$

zero-or-successor rule

Let X be a set of variables. We set

$$\Theta(X) := \{\theta : X \to \mathbb{N}[V] \mid \text{ for all } x \in X : \theta(x) \in \{0, x+1\}\}$$

Let $\mathsf{vars}(p)$ be the set of variables that occur in $p \in \mathbb{Z}[V]$. We translate $B_1^{\exists}$ into an inference rule

$$\frac{p\theta \text{ for all } \theta \in \Theta(\mathsf{vars}(p))}{p}\ \textit{zero-or-successor}$$

where p is not signed.

Calculus $\mathcal{AB}$

Example

Let $\mathcal{AB}$ be the proof calculus operating on $\mathbb{Z}[V]$ with the inference rules *signed* and *zero-or-successor*.
We abbreviate *signed* as s and *zero-or-successor* as z:

$$\dfrac{\dfrac{}{1}\,s \quad \dfrac{}{-2y-1}\,s \quad \dfrac{}{-2x-1}\,s \quad \dfrac{\dfrac{}{-1}\,s \quad \dfrac{}{-1}\,s \quad \dfrac{}{-1}\,s \quad \dfrac{}{2xy+2x+2y+1}\,s}{2xy-1}\,z}{2xy-2x-2y+1}\,z$$

Theorem (Soundness and Completeness of $\mathcal{AB}$)

$AB \vdash \forall \bar{x}\, t \neq u$ if and only if $\mathcal{AB} \vdash \mathrm{poly}(t) - \mathrm{poly}(u)$

Proof sketch.

Do proof translations in both directions. $\square$

Outline

Tilted polynomials

Definition (tilted polynomial)

We say $p \in \mathbb{Z}[V]$ is *positively tilted* if for all monomials m^- with $[m^-]p^- \neq 0$ there exists a monomial m^+ with $[m^+]p^+ \neq 0$ such that m^- strictly divides m^+.

We say p is *negatively tilted*, if $-p$ is positively tilted.

If p is positively or negatively tilted, we say p is *tilted*.

Example

$$\left.\begin{aligned} x^2 - x + 1 \\ xy - 2x - 2y \end{aligned}\right\} \text{ tilted}$$

$$\left.\begin{aligned} 0 \\ x - y \\ xy - x^2 - y^2 \end{aligned}\right\} \text{ not tilted}$$

Closure property in $\mathcal{AB}$

Lemma

If $p \in \mathbb{Z}[V]$ is positively (negatively) signed, then p is positively (negatively) tilted.

Lemma

Let $p \in \mathbb{Z}[V]$ and $\theta(x) := x + 1$. Then p is positively (negatively) tilted if and only if $p\theta$ is positively (negatively) tilted.

Corollary

If $\mathcal{AB} \vdash p$, then p is tilted.

An order on $\mathbb{N}[V]$

- For $p \in \mathbb{N}[V]$ we write $\text{mons}(p)$ for the multiset of monomials where each monomial m occurs $[m]p$ many times.
- For $p, q \in \mathbb{N}[V]$ we write $p <_{mon} q$ if for all $m \in \text{mons}(p) - \text{mons}(q)$ there exists an $m' \in \text{mons}(q) - \text{mons}(p)$ such that m strictly divides m'.
- Note: $p \in \mathbb{Z}[V]$ is positively (negatively) tilted if and only if $p^+ >_{mon} p^-$ ($p^- >_{mon} p^+$).
- $<_{mon}$ is the multiset extension of strict divisibility of monomials.

Lemma

$<_{mon}$ is a well-founded partial order on $\mathbb{N}[V]$.

An order on tilted polynomials

- For $p, q \in \mathbb{Z}[V]$ we write $p \prec_{vars} q$ if $|\text{vars}(p)| < |\text{vars}(q)|$.
- For tilted p we set $\min(p) := \min_{<_{mon}}(p^+, p^-)$.
- For tilted p, q we write $p \prec_{mon} q$ if $\min(p) <_{mon} \min(q)$.
- Let $\prec_t$ to be the lexicographic product $\prec_{vars} \times \prec_{mon}$.

Lemma

$\prec_{vars}$, $\prec_{mon}$ and $\prec_t$ are well-founded partial orders on tilted polynomials.

Proof candidate trees

Definition

For $p \in \mathbb{Z}[V]$ we recursively define the *proof candidate tree of* p as the smallest tree $T(p)$ such that

- p is a node of $T(p)$ and
- if q is a node of $T(p)$, q is tilted and not signed, then $T(p)$ contains all nodes $q\theta$ for $\theta \in \Theta(\mathrm{vars}(q))$. In that case $(q, q\theta)$ is an edge of $T(p)$.

Proof candidate trees

Finiteness of $T(p)$

Lemma

$T(p)$ is finitely branching.

Lemma

Let $p \in \mathbb{Z}[V]$ be tilted and let $\theta \in \Theta(\text{vars}(p))$. Then $p \succ_t p\theta$.

Proposition

$T(p)$ is finite.

Proof.

We use Kőnig's lemma:

- $T(p)$ is finitely branching.
- If a branch in $T(p)$ only contains tilted polynomials, then it is well-ordered by $\prec_t$ which means it is finite.
- If a branch in $T(p)$ contains a non-tilted polynomial, the branch must be finite since no edges originate from non-tilted polynomials.

□

Decision procedure

Lemma

$\mathcal{AB} \vdash p$ if and only if all leaves of $T(p)$ are signed polynomials.

Corollary

$\mathcal{AB}$ is decidable.

Decision procedure.

Construct $T(p)$ and check if all leaves are signed polynomials. $\square$

Calculus $\mathcal{ABC}$

Consider the additional axiom C, the universal closure of

$$x \neq 0 \to (x \cdot y = x \cdot z \to y = z)$$

Over AB, it is equivalent to the universal closure of

$$y \neq z \to \mathsf{s}(x) \cdot y \neq \mathsf{s}(x) \cdot z.$$

This translates into the inference rule

$$\frac{q}{pq}\ \textit{factor}$$

where p is signed.

Let $\mathcal{ABC}$ be the proof calculus consisting of the rules from $\mathcal{AB}$ and the additional rule *factor*.

Theorem (Soundness and Completeness of $\mathcal{ABC}$)

$\mathcal{ABC} \vdash \mathrm{poly}(t) - \mathrm{poly}(u)$ *if and only if* $ABC \vdash \forall \bar{x}\, t \neq u$

Equivalence of $\mathcal{AB}$ and $\mathcal{ABC}$

Lemma

If p is signed and θ is a substitution, then $p\theta$ is signed.

Lemma

If p and q are signed, then pq is signed.

Proposition

$\mathcal{ABC} \vdash p$ *if and only if* $\mathcal{AB} \vdash p$.

Proof sketch for $\Rightarrow$.

- Move instances of *factor* above instances of *zero-or-successor* (uses that signed polynomials are closed under substitution).
- Top-most chains of *factor* inferences can be replaced by a single *signed* using previous lemma.

$\square$

Corollary

$D_{AB} = D_{ABC}$

Outline

Summary

Main Theorem

D_{IOp} is decidable.

Proof sketch.

- By result from Schmerl it suffices to prove decidability of D_{AB}
- Construct a specialized proof calculus $\mathcal{AB}$ operating on $\mathbb{Z}[V]$.
- Show soundness and completeness with respect to disequalities using proof-theoretic methods.
- Show that $\mathcal{AB}$ is decidable with closure properties and an appropriate well-order.

$\square$

Outlook

Theory T	D_T decidable?
Q	yes[7]
IOp, AB, ABC_d, ABC	yes
PA^-	unknown
IOpen	unknown
extensions of IU_1^-	no[8]
extensions of $I\Delta_0 + EXP$	no[9]

[7] Jeř16.

[8] Kay93.

[9] GD82.

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