Formula equations and the affine solution problem

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Motivation

- Generalizes many problems in computational logic such as software verification and inductive theorem proving.
- Serves as a common logical language for these disparate research activities.
- Studied since end of 19th/beginning of 20th century
- No thorough investigation yet

Outline

Short introduction to mathematical logic

Formula equations

Affine solution problem

Convex solution problem

Second-order logic

Syntax

- *Language*: Set of function/relation symbols with assigned arities, e.g.
  - groups: $\{\cdot/2, e/0, {}^{-1}/1, = /2\}$
  - arithmetic: $\{+/2, \cdot/2, 0/0, 1/0, = /2, \leq /2\}$
  - graphs: $\{E/2, = /2\}$
- *Terms*: Built inductively from variables $u, v, w, \ldots$ and function symbols, e.g.
  - $u \cdot (v \cdot e)$
  - $(1 + 1) + (0 \cdot u)$

Second-order logic

Syntax

- *Formulas*: Built inductively from
  - $\top$, $\bot$ ("true", "false")
  - $R(t_1, \ldots, t_n)$ for relation symbol R of arity n and terms $t_1, \ldots, t_n$
  - $X(t_1, \ldots, t_n)$ for relation variable X of arity n and terms $t_1, \ldots, t_n$
  - propositional connectives: negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), implication ($\rightarrow$), equivalence ($\leftrightarrow$).
  - quantification over individuals: for all ($\forall u$) and exists ($\exists u$)
  - quantification over relations: for all ($\forall X$), exists ($\exists X$)
  - e.g.
    - $\neg\bot \vee \top$
    - $A(x) \wedge \neg A(x)$
    - $\exists X\, \forall u\, \forall v\, (X(u, v) \rightarrow X(v, u))$
- A formula is called *first-order* if it does not contain quantification over relations.

Second-order logic

Semantics

- *L-Structure* $\mathcal{M}$: A non-empty set M together with
  - assignment of function symbols f/n of L to functions $f^{\mathcal{M}} : M^n \to M$.
  - assignment of relation symbols R/n of L to relations $R^{\mathcal{M}} \subseteq M^n$.
  - e.g. $\mathbb{N}$ together with the assignment of $+$, $\cdot$, 0, 1, $=$, $\leq$ to the actual addition, multiplication, zero, one, equality and less-than-or-equal on natural numbers
- We write $\mathcal{M} \models \varphi$ ("$\mathcal{M}$ satisfies/models φ") if formula φ holds in $\mathcal{M}$, e.g.
  - $\mathbb{N} \models \forall u\, 0 \leq u$, $\mathbb{N} \not\models \exists u \forall v\, v \leq u$
  - $\mathbb{C} \models \forall u \forall v\, u \cdot v = v \cdot u$, but $\mathbb{H} \not\models \forall u \forall v\, u \cdot v = v \cdot u$
- We write $\models \varphi$ ("φ is valid") if φ holds in all L-structures, e.g.
  - $\models (A \land B) \to A$
  - $\models \forall X\, (\exists u \forall v\, X(u, v) \to \forall v \exists u\, X(u, v))$

Decidability

- *Decision problem*: Let an input set S be fixed. Given $D \subseteq S$, is there an algorithm which given an input $x \in S$ answers "yes" if $x \in D$ and answers "no" if $x \notin D$?
- If such an algorithm exists, we call D *decidable*.
- Examples of decidable sets:
  - the set of prime numbers inside the natural numbers,
  - the set of connected graphs inside the class of finite graphs.
- Examples of undecidable sets:
  - The set of programs p and inputs i such that p terminates on input i (halting set),
  - the set of first-order formulas φ such that $\mathbb{N} \models \varphi$.

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Formula equations

Motivation

- Equation: Given terms $t(x)$, $s(x)$, find an a such that

$$t(a) = s(a).$$

- On formulas: Given formulas $\varphi(X)$, $\psi(X)$, find a formula χ such that

$$\varphi(\chi) = \psi(\chi)$$

$$\models \varphi(\chi) \leftrightarrow \psi(\chi)$$

- $\varphi(\chi)$ describes the formula where every occurrence of $X(t_1, \ldots, t_n)$ in φ is substituted by $\chi(t_1, \ldots, t_n)$ (given compatible arities)
- Simplification: Given $\varphi(X)$, find a formula χ such that

$$\models \varphi(\chi)$$

Formula equations

Definition

A *formula equation* is a formula of the form $\exists X_1 \dots X_n \, \varphi(X_1, \dots, X_n)$ where $\varphi(X_1 \dots X_n)$ is a first-order formula. A *solution of* $\exists X_1 \dots X_n \, \varphi(X_1, \dots, X_n)$ *modulo a structure* $\mathcal{M}$ is a tuple of first-order formulas $\chi_1, \dots, \chi_n$ such that $\mathcal{M} \models \varphi(\chi_1, \dots, \chi_n)$.

- We often abbreviate tuples $(a_1, \dots, a_n)$ by $\overline{a}$
  - formula equation $\exists \overline{X} \, \varphi(\overline{X})$
  - solution tuple $\overline{\chi}$

Formula equations

Examples

- $\exists X \, X(0)$
  - some solutions are $\chi(u) := u = 0$ and $\chi(u) := \top$ (modulo any structure $\mathcal{M}$)
- $\exists X \, (X(0) \wedge \neg X(0))$
  - has no solutions
- $\exists X \, (X(0) \wedge \forall u \, (X(u) \rightarrow X(u+2)) \wedge \forall u \, (X(u) \rightarrow \neg X(u+1)))$
  - a solution modulo $\mathbb{N}$ is $\chi(u) := \exists v \, u = v + v$

Remark

There are valid, but unsolvable formula equations, i.e. there exist $\varphi(X)$ such that $\mathbb{N} \models \exists X \, \varphi(X)$, but $\exists X \, \varphi(X)$ has no solutions modulo $\mathbb{N}$.

Solution problems

- Is there an algorithm, such that given any instance of a formula equation, determines whether it has a solution modulo a structure $\mathcal{M}$?

Instances of solution problems capture the following problems:

- Satisfiability of propositional formulas (NP-complete)
- Validity of first-order formulas (undecidable)
- Software verification (undecidable)
- Affine solution problem (decidable)
- Convex solution problem (open)

Other avenues for research:

- Use techniques from one area and apply it to solve problems in another area
- Find sets of formulas which are closed under solutions to formula equations
- ...

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Affine formula equations

We work in the language L_{aff}:

- zero $0/0$,
- one $1/0$,
- addition $+/2$,
- equality $= /2$ and
- scalar multiplication ($c/1$ for every $c \in \mathbb{Q}$).

modulo the theory of the rational numbers $\mathbb{Q}$.

- Can assume w.l.o.g. that every term $t(x_1, \ldots, x_n)$ is of the form $c_0 + \sum_{i=1}^{n} c_i x_i$ and every first-order atomic formula is of the form $t(x_1, \ldots, x_n) = 0$ for some term t
- Every term $t(x_1, \ldots, x_n)$ induces an affine function $t^{\mathbb{Q}} : \mathbb{Q}^n \to \mathbb{Q}$
- Every first-order atomic formula $A(x_1, \ldots, x_n)$ induces an affine subspace $A^{\mathbb{Q}} \subseteq \mathbb{Q}^n$ or the empty set
- Conjunctions of atomic formulas describe systems of linear equations and thus affine spaces.

Affine formula equations

Definition (Affine solution problem)

Input: A quantifier-free L_{aff}-formula $\varphi(\overline{X}, \overline{u})$
Output: Is there a solution $\overline{\chi}$ of $\exists \overline{X}\, \forall \overline{u}\, \varphi(\overline{X}, \overline{u})$ modulo $\mathbb{Q}$ such that all χ_i are conjunctions of atoms?

Theorem (Hetzl, Zivota '19)

The affine solution problem is decidable.[1]

[1] HZ19.

Decision procedure

Clausification

Every quantifier-free formula can transformed into an equivalent formula φ' in *conjunctive normal form*:

$$C_1 \wedge \cdots \wedge C_k$$

where each C_i is a *clause*, i.e. of the form

$$A(\overline{u}) \wedge X_{i_1}(\overline{t_1}(\overline{u})) \wedge \cdots \wedge X_{i_l}(\overline{t_l}(\overline{u})) \\ \to B_1(\overline{u}) \vee \cdots \vee B_m(\overline{u}) \vee X_{i_{l+1}}(\overline{t_{l+1}}(\overline{u})) \vee \ldots X_{i_{l+r}}(\overline{t_{l+r}}(\overline{u}))$$

Decision procedure
Example

Consider the affine formula equation

$$\exists X\,\forall u\,\forall v\,\begin{pmatrix} X(1,0) \\ \wedge(X(-u,v) \to X(-v,u) \vee X(u,-v)) \\ \wedge(X(u,v) \to u = v \vee v = 0) \end{pmatrix}$$

which is already in clause form. Its clauses are

$$\begin{aligned} &\to X(1,0) \\ X(-u,v) &\to X(-v,u) \vee X(u,-v) \\ X(u,v) &\to u = v \vee v = 0. \end{aligned}$$

Decision procedure

Translation to affine conditions

Over $\mathbb{Q}$ the clause

$$A(\overline{u}) \wedge X_{i_1}(\overline{t_1}(\overline{u})) \wedge \cdots \wedge X_{i_l}(\overline{t_l}(\overline{u})) \\ \rightarrow B_1(\overline{u}) \vee \cdots \vee B_m(\overline{u}) \vee X_{i_{l+1}}(\overline{t_{l+1}}(\overline{u})) \vee \ldots X_{i_{l+r}}(\overline{t_{l+r}}(\overline{u}))$$

translates into the condition

$$A^{\mathbb{Q}} \cap \bigcap_{j=1}^{l} (\overline{t_j}^{\mathbb{Q}})^{-1}(\mathcal{X}_{i_j}) \subseteq \bigcup_{k=1}^{m} B_k^{\mathbb{Q}} \cup \bigcup_{k=l+1}^{l+r} (\overline{t_k}^{\mathbb{Q}})^{-1}(\mathcal{X}_{i_k})$$

where the $\mathcal{X}_i$ are unknown affine subspaces.

Affine solution problem

Geometric formulation

Input: $p \in \mathbb{N}$ and for each $1 \leq i \leq m$ affine spaces $\mathcal{A}^i, \mathcal{B}^i_1, \ldots, \mathcal{B}^i_{s_i} \subseteq \mathbb{Q}^n$, affine transformations $T^i_1, \ldots, T^i_{l_i}, \ldots T^i_{l_i+r_i}$ and indices for the unknowns $j_1, \ldots, j_{l_i}, \ldots, j_{l_i+r_i} \in \{1, \ldots, p\}$.
Output: For all $1 \leq i \leq m$ are there affine spaces $\mathcal{X}_1, \ldots, \mathcal{X}_p \subseteq \mathbb{Q}^n$ such that for all $1 \leq i \leq m$ there holds

$$\mathcal{A}^i \cap \bigcap_{k=1}^{l_i} (T^i_k)^{-1}(\mathcal{X}_{j_k}) \subseteq \bigcup_{k=1}^{s_i} \mathcal{B}^i_k \cup \bigcup_{k=l_i+1}^{l_i+r_i} (T^i_k)^{-1}(\mathcal{X}_{j_k})?$$

Decision procedure

Example

Remember the clauses

$$\begin{aligned} &\to X(1,0) \\ X(-u,v) &\to X(-v,u) \vee X(u,-v) \\ X(u,v) &\to u = v \vee v = 0. \end{aligned}$$

Let $f(u,v) := (1,0)^T$, $g(u,v) := (-u,v)^T$, $h(u,v) := (-v,u)^T$, $k(u,v) := (u,-v)^T$. The clauses translate into conditions on the affine space $\mathcal{X}$:

$$\begin{aligned} \mathbb{Q}^2 &\subseteq f^{-1}(\mathcal{X}) \\ g^{-1}(\mathcal{X}) &\subseteq h^{-1}(\mathcal{X}) \cup k^{-1}(\mathcal{X}) \\ \mathcal{X} &\subseteq \left[(1,1)^T\right] \cup \left[(1,0)^T\right] \end{aligned}$$

Decision procedure

Covering property

Lemma

Let V be a vector space over $\mathbb{Q}$ and let $\mathcal{A}, \mathcal{B}_1, \ldots, \mathcal{B}_m$ be affine subspaces of V. If $\mathcal{A} \subseteq \bigcup_{i=1}^{m} \mathcal{B}_i$, then $\mathcal{A} \subseteq \mathcal{B}_i$ for some $i \in \{1, \ldots, m\}$.

Decision procedure

Projections

Definition

A *projection* of an affine condition of the form

$$A^{\mathbb{Q}} \cap \bigcap_{j=1}^{l} (\overline{t_j}^{\mathbb{Q}})^{-1}(\mathcal{X}_{i_j}) \subseteq \bigcup_{k=1}^{m} B_k^{\mathbb{Q}} \cup \bigcup_{j=l+1}^{l+r} (\overline{t_j}^{\mathbb{Q}})^{-1}(\mathcal{X}_{i_j})$$

is a condition of the form

$$A^{\mathbb{Q}} \cap \bigcap_{j=l+1}^{l+r} (\overline{t_j}^{\mathbb{Q}})^{-1}(\mathcal{X}_{i_j}) \subseteq B_k^{\mathbb{Q}} \quad \text{(upper bound condition)}$$

or

$$A^{\mathbb{Q}} \cap \bigcap_{j=l+1}^{l+r} (\overline{t_j}^{\mathbb{Q}})^{-1}(\mathcal{X}_{i_j}) \subseteq (\overline{t_k}^{\mathbb{Q}})^{-1}(\mathcal{X}_{i_k}) \overline{t_k}^{\mathbb{Q}}(A^{\mathbb{Q}} \cap \bigcap_{j=1}^{l} (\overline{t_j}^{\mathbb{Q}})^{-1}(\mathcal{X}_{i_j})) \subseteq \mathcal{X}_{i_k} \quad (\text{l}$$

Decision procedure

Projections

Corollary

An affine condition of the form

$$A^{\mathbb{Q}} \cap \bigcap_{j=1}^{l} (\overline{t_j}^{\mathbb{Q}})^{-1}(\mathcal{X}_{i_j}) \subseteq \bigcup_{k=1}^{m} B_k^{\mathbb{Q}} \cup \bigcup_{j=l+1}^{l+r} (\overline{t_j}^{\mathbb{Q}})^{-1}(\mathcal{X}_{i_j})$$

is solvable iff one of its projections is solvable.

Decision procedure

Example

Remember the affine conditions:

$$f(\mathbb{Q}^2) \subseteq \mathcal{X}$$
$$g^{-1}(\mathcal{X}) \subseteq h^{-1}(\mathcal{X}) \cup k^{-1}(\mathcal{X})$$
$$\mathcal{X} \subseteq \left[(1,1)^T\right] \cup \left[(1,0)^T\right]$$

Induces four sets of affine conditions that are projections:

$$f(\mathbb{Q}^2) \subseteq \mathcal{X} \quad h(g^{-1}(\mathcal{X})) \subseteq \mathcal{X} \quad \mathcal{X} \subseteq \left[(1,1)^T\right]$$

$$f(\mathbb{Q}^2) \subseteq \mathcal{X} \quad h(g^{-1}(\mathcal{X})) \subseteq \mathcal{X} \quad \mathcal{X} \subseteq \left[(1,0)^T\right]$$

$$f(\mathbb{Q}^2) \subseteq \mathcal{X} \quad k(g^{-1}(\mathcal{X})) \subseteq \mathcal{X} \quad \mathcal{X} \subseteq \left[(1,1)^T\right]$$

$$f(\mathbb{Q}^2) \subseteq \mathcal{X} \quad k(g^{-1}(\mathcal{X})) \subseteq \mathcal{X} \quad \mathcal{X} \subseteq \left[(1,0)^T\right]$$

Decision procedure

Fixed-point iteration

Define

$$\mathcal{X}_i^{(0)} := \emptyset$$
$$\mathcal{X}_i^{(j+1)} := \mathit{aff}(\mathcal{X}_i^{(j)} \cup \mathcal{Y}_i^{(j)})$$

where $\mathcal{Y}_i^{(j)}$ is the union of left-hand sides of lower bound conditions of $\mathcal{X}_i$ where $\mathcal{X}_i$ is substituted by $\mathcal{X}_i^{(j)}$.

Theorem

The affine solution problem is decidable.

Proof.

- Fixed point iteration is monotone.
- Affine spaces satisfy ascending chain condition.
- Iteration terminates with *smallest* solution of lower bound conditions.
- Suffices to check if upper bound conditions are satisfied.

□

Decision procedure

Example

Remember the third set of projections:

$$\begin{aligned} f(\mathbb{Q}^2) &\subseteq \mathcal{X} \\ k(g^{-1}(\mathcal{X})) &\subseteq \mathcal{X} \\ \mathcal{X} &\subseteq \left[(1,1)^T\right] \end{aligned}$$

Doing the fixed point iteration yields

$$\begin{aligned} &\mathcal{X}^{(0)} = \emptyset \\ &\mathcal{X}^{(1)} = aff(f(\mathbb{Q}^2)) = \left\{(1,0)^T\right\} \\ &\mathcal{X}^{(2)} = aff(\mathcal{X}^{(1)} \cup k(g^{-1}(\mathcal{X}^{(1)}))) = \left[(1,0)^T\right] \\ &\mathcal{X}^{(3)} = \ldots = \left[(1,0)^T\right] \end{aligned}$$

Fixed point reached, but does not satisfy upper bound.

Decision procedure

Example

Now instead consider the fourth set of projections:

$$\begin{aligned} f(\mathbb{Q}^2) &\subseteq \mathcal{X} \\ k(g^{-1}(\mathcal{X})) &\subseteq \mathcal{X} \\ \mathcal{X} &\subseteq \left[(1,0)^T\right] \end{aligned}$$

Same lower bound conditions as before thus fixed point iteration yields the same result $\left[(1,0)^T\right]$ which this time satisfies the upper bound.

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Convex solution problem

Convex solution problem

We work in the language L_{conv}:

- zero $0/0$,
- one $1/0$,
- addition $+/2$,
- equality $=/2$,
- scalar multiplication ($c/1$ for every $c \in \mathbb{Q}$) and
- **inequality** $\leq /2$.

modulo the theory of the rational numbers $\mathbb{Q}$.

- Can assume w.l.o.g. that every first-order atomic formula is of the form $\sum_{i=1}^{n} c_i x_i \leq d$
- Conjunctions of atomic formulas describe systems of linear inequalities and thus (possibly unbounded) *convex polytopes*.

Convex solution problem

Geometric formulation

Definition (Convex solution problem)

Input: $p \in \mathbb{N}$ and for each $1 \leq i \leq m$ **convex polytopes** $\mathcal{A}^i, \mathcal{B}^i_1, \ldots, \mathcal{B}^i_{s_i} \subseteq \mathbb{Q}^n$, affine transformations $T^i_1, \ldots, T^i_{l_i}, \ldots T^i_{l_i+r_i}$ and indices for the unknowns $j_1, \ldots, j_{l_i}, \ldots, j_{l_i+r_i} \in \{1, \ldots, p\}$.
Output: For all $1 \leq i \leq m$ are there **convex polytopes** $\mathcal{X}_1, \ldots, \mathcal{X}_p \subseteq \mathbb{Q}^n$ such that for all $1 \leq i \leq m$ there holds

$$\mathcal{A}^i \cap \bigcap_{k=1}^{l_i} (T^i_k)^{-1}(\mathcal{X}_{j_k}) \subseteq \bigcup_{k=1}^{s_i} \mathcal{B}^i_k \cup \bigcup_{k=l_i+1}^{l_i+r_i} (T^i_k)^{-1}(\mathcal{X}_{j_k})?$$

Decidability is still open! Previous proof fails because:

- Covering property fails for convex polytopes
- Iteration procedure might not terminate (no ACC)

Convex solution problem

In the rational plane

Definition (Convex solution problem in the rational plane)

Input: A rotation $T : \mathbb{Q}^2 \to \mathbb{Q}^2$, and points $p, q \in \mathbb{Q}^2$
Output: Is there a convex polytope $\mathcal{X}$ such that $p \in \mathcal{X}, q \notin \mathcal{X}$ and $T(\mathcal{X}) \subseteq \mathcal{X}$?

Theorem (Zivota '21)

The convex solution problem in the rational plane is decidable.[2]

[2] Ziv21.

Convex solution problem

Intervals

Definition (Box)

A *box* $B \subseteq \mathbb{Z}^n$ is a product of (possibly unbounded) intervals in $\mathbb{Z}$.

Definition (Interval solution problem)

Input: $p \in \mathbb{N}$ and for each $1 \leq i \leq m$ **boxes** $\mathcal{A}^i, \mathcal{B}^i_1, \dots, \mathcal{B}^i_{s_i} \subseteq \mathbb{Z}^n$, affine transformations $T^i_1, \dots, T^i_{l_i}, \dots T^i_{l_i+r_i}$ and indices for the unknowns $j_1, \dots, j_{l_i}, \dots, j_{l_i+r_i} \in \{1, \dots, p\}$.
Output: For all $1 \leq i \leq m$ are there **boxes** $\mathcal{X}_1, \dots, \mathcal{X}_p \subseteq \mathbb{Z}^n$ such that for all $1 \leq i \leq m$ there holds

$$\mathcal{A}^i \cap \bigcap_{k=1}^{l_i} (T^i_k)^{-1}(\mathcal{X}_{j_k}) \subseteq \bigcup_{k=1}^{s_i} \mathcal{B}^i_k \cup \bigcup_{k=l_i+1}^{l_i+r_i} (T^i_k)^{-1}(\mathcal{X}_{j_k})?$$

Theorem (Zivota '21)

The interval solution problem is decidable.[3]

[3] Ziv21.

Generalized convex solution problem

We work in the language L_{conv}:

- zero $0/0$,
- one $1/0$,
- addition $+/2$,
- equality $=/2$,
- **multiplication** $\cdot/2$ and
- inequality $\leq/2$.

modulo the theory of the rational numbers $\mathbb{Q}$.

- Can assume w.l.o.g. that every first-order atomic formula is of the form $p(x_1, \ldots, x_n) \leq 0$ for a polynomial $p \in \mathbb{Q}[x_1, \ldots, x_n]$
- Conjunctions of atomic formulas describe systems of polynomial inequalities.

Generalized convex solution problem

Definition

We call a set $\mathcal{A} \subseteq \mathbb{Q}^n$ *polynomially constrained* if it is a finite intersection of sets of the form $\{\overline{x} \in \mathbb{Q}^n \mid p(\overline{x}) \leq 0\}$ for a polynomial $p \in \mathbb{Q}[\overline{x}]$.

Definition (Generalized convex solution problem)

Input: $p \in \mathbb{N}$ and for each $1 \leq i \leq m$ **polynomially constrained** $\mathcal{A}^i, \mathcal{B}^i_1, \ldots, \mathcal{B}^i_{s_i} \subseteq \mathbb{Q}^n$, affine transformations $T^i_1, \ldots, T^i_{l_i}, \ldots T^i_{l_i+r_i}$ and indices for the unknowns $j_1, \ldots, j_{l_i}, \ldots, j_{l_i+r_i} \in \{1, \ldots, p\}$.
Output: For all $1 \leq i \leq m$ are there **convex polytopes** $\mathcal{X}_1, \ldots, \mathcal{X}_p \subseteq \mathbb{Q}^n$ such that for all $1 \leq i \leq m$ there holds

$$\mathcal{A}^i \cap \bigcap_{k=1}^{l_i} (T^i_k)^{-1}(\mathcal{X}_{j_k}) \subseteq \bigcup_{k=1}^{s_i} \mathcal{B}^i_k \cup \bigcup_{k=l_i+1}^{l_i+r_i} (T^i_k)^{-1}(\mathcal{X}_{j_k})?$$

Generalized convex solution problem

Theorem (Monniaux '19)

The generalized convex solution problem is undecidable.[4]

In fact it suffices to only allow the polynomials in the input to be quadratic!

[4] Mon19.

Conclusion

- Formula equations serve as a common framework for many problems in computational logic
  - SAT problem, inductive theorem proving, software verification, ...
- Potential to integrate techniques from disparate research communities
- Solution problems suggest lots of avenues for further research

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